$$\hat{\beta}_0 = \bar{Y} - \hat{\beta}_1 \bar{X}$$

$$Var(\hat{\beta}_0) = Var(\bar{Y}) + \bar{X}^2 Var(\hat{\beta}_1) - 2\bar{X}\, \underbrace{Cov(\bar{Y}, \hat{\beta}_1)}$$

$$S_{xx} = \Sigma (x_i - \bar{x})^2$$

$$C_i = \frac{x_i - \bar{x}}{S_{xx}}$$

$$Cov(\bar{Y}, \hat{\beta}_1) = Cov\left(\Sigma \tfrac{1}{n} Y_i,\ \Sigma \frac{(x_i - \bar{x}) Y_i}{S_{xx}}\right)$$

$$= Cov\left(\Sigma \tfrac{1}{n} Y_i,\ \Sigma c_i Y_i\right) \Rightarrow \sum_i \sum_j \tfrac{1}{n} c_j\, Cov(Y_i, Y_j)$$

$Cov(Y_i, Y_j) = 0 \quad Y_i \neq Y_j \quad Y_i \perp Y_j$ or $i \neq j$

$Cov(Y_i, Y_i) = Var(Y_i) \quad Y_i = Y_j$ or $i = j$

$$= \tfrac{1}{n} \sum_i C_i\, Var(Y_i) \Rightarrow \frac{\sigma^2}{n} \Sigma C_i \Rightarrow \Sigma C_i = 0 \Rightarrow Cov(\bar{Y}, \hat{\beta}_1) = 0$$

$$Var(\hat{\beta}_0) = Var(\bar{Y}) + \bar{X}^2\, Var(\hat{\beta}_1) \qquad Var(\hat{\beta}_1) = \frac{\sigma^2}{S_{xx}}$$

$$= \frac{\sigma^2}{n} + \frac{\bar{x}^2 \sigma^2}{S_{xx}} \Rightarrow \sigma^2 \left(\frac{1}{n} + \frac{\bar{x}^2}{S_{xx}}\right)$$

$$= \sigma^2 \left(\frac{S_{xx}}{n S_{xx}} + \frac{n\bar{x}^2}{n S_{xx}}\right) \Rightarrow \sigma^2 \left(\frac{\Sigma (x_i - \bar{x})^2 + n\bar{x}^2}{n S_{xx}}\right)$$

$$= \sigma^2 \left(\frac{\Sigma (x_i^2 - 2x_i\bar{x} + \bar{x}^2) + n\bar{x}^2}{n S_{xx}}\right) \Rightarrow \sigma^2 \frac{\left(\Sigma x_i^2 \cancel{- 2n\bar{x}^2} \cancel{+ n\bar{x}^2} \cancel{+ n\bar{x}^2}\right)}{n S_{xx}}$$

$$Var(\hat{\beta}_0) = \frac{\sigma^2 \Sigma x_i^2}{n S_{xx}} \qquad \hat{Var}(\hat{\beta}_0) = \frac{\hat{\sigma}^2 \Sigma x_i^2}{n S_{xx}} \qquad SE(\hat{\beta}_0) = \sqrt{\frac{\hat{\sigma}^2 \Sigma x_i^2}{n S_{xx}}}$$