Review:

More Probability Theory

- **Continuous Random Variables**
- Uniform Distribution
- Normal Distribution
- Moment Generating Functions
- Characteristic Functions
- Poisson Distribution
- Binomial Distribution
- Uniform Distribution
- Normal Distribution
- MGF Properties

Continuous Random Variables

A random variable X is considered continuous if the $P(X = x)$ does not exist. $= 0$

$$P(a < X < b) = \int_a^b f(x)\,dx$$

↑ Probability density function

$$P(X=a) = P(a < X < a)$$

$$\int_a^a f(x)\,dx = F(a) - F(a) = 0$$

$\uparrow$

Antiderivative

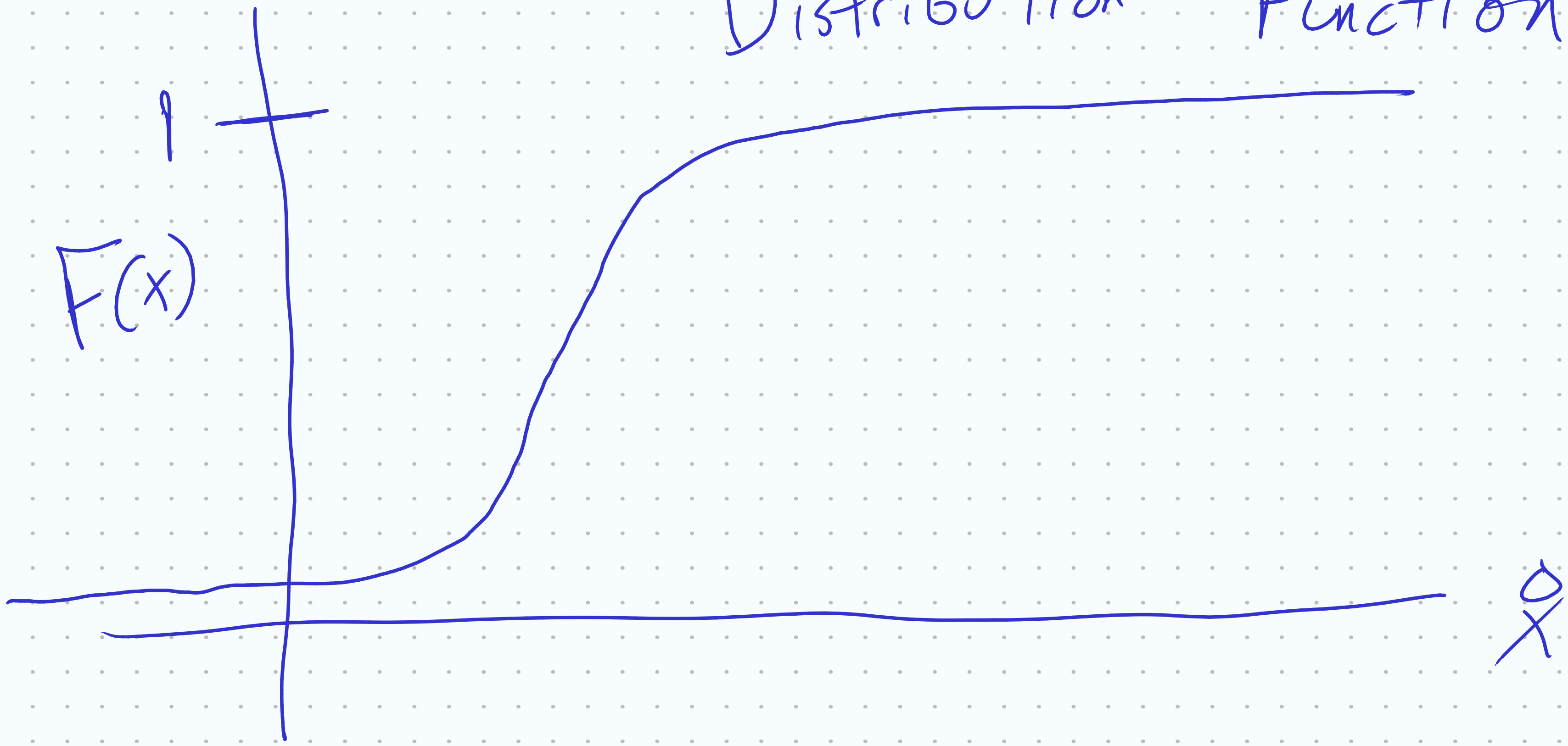
CDF

The cumulative distribution function of X provides the $P(X \leq x)$, denoted by $F(x)$, for the domain of X.

Properties of the CDF of X:

1. $F(-\infty) \equiv \lim_{y \to -\infty} F(y) = 0$
2. $F(\infty) \equiv \lim_{y \to \infty} F(y) = 1$
3. $F(x)$ is a nondecreaseing function

Distribution Function



PDF

The probability density function of the random variable X is given by

$$f(x) = \frac{dF(x)}{d(x)} = F'(x)$$

wherever the derivative exists.

Properties of pdfs:

$1 > f(x) > 0$

$\int_x f(x)\,dx = 1$

Expected Value

The expected value for a continuous distribution is defined as

$$E(X) = \int_X x f(x) dx$$

The expectation of a function $g(X)$ is defined as

$$E\{g(X)\} = \int_X g(x) f(x) dx$$

$E(x)$	1	2	3	4	5
$f(x)$	0.1	0.2	0.3	0.2	0.2

$$\int x \, f(x)$$

$$1(0.1) + 2(0.2) + 3(0.3) + 4(0.2) + 5(0.2)$$

$$E(x^2) \qquad E\left(\frac{x^2}{3}\right)$$

Expected Value Properties

1. $E(c) = c$, where c is constant
2. $E\{cg(X)\} = cE\{g(X)\}$
3. $E\{g_1(X) + g_2(X) + \cdots + g_n(X)\} = E\{g_1(X)\} + E\{g_2(X)$

$-\infty < X < \infty$ $f(x)$ -

$$E(2) = \int_{-\infty}^{\infty} 2 f(x)\, dx$$

$$= 2 \int_{-\infty}^{\infty} f(x)\, dx$$

$$= 2$$

$0 < X < \infty$ $f(x)$ $E(x^2) = \int_0^\infty x^2 f(x)\, dx$

$Y = 2X^2$ $E(Y)$

$$E(2X^2) = \int_0^\infty 2x^2 f(x)\, dx$$

$$= 2\int_0^\infty x^2 f(x)\, dx$$

$$= 2E(X^2)$$

$0 < X < \infty$ $\qquad f(x)$

$g_1(x) + g_2(x) + g_3(x)$

$Y = X + X^2 + X^3$

$E(Y) = E(X + X^2 + X^3)$

$\int_0^\infty (X + X^2 + X^3) f(x)\, dx$

$\int_0^\infty X f(x) + X^2 f(x) + X^3 f(x)\, dx$

$\int_0^\infty X f(x)\, dx + \int_0^\infty X^2 f(x)\, dx + \int_0^\infty X^3 f(x)\, dx$

$E(X) + E(X^2) + E(X^3)$

Variance

The variance of continuous variable is defined as

$$Var(X) = E[\{X - E(X)\}^2] = \int \{X - E(X)\}^2 f(x)dx$$

$E(X^2) - E(X)^2$

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Uniform Distribution

A random variable is said to follow uniform distribution if the density function is constant between two parameters.

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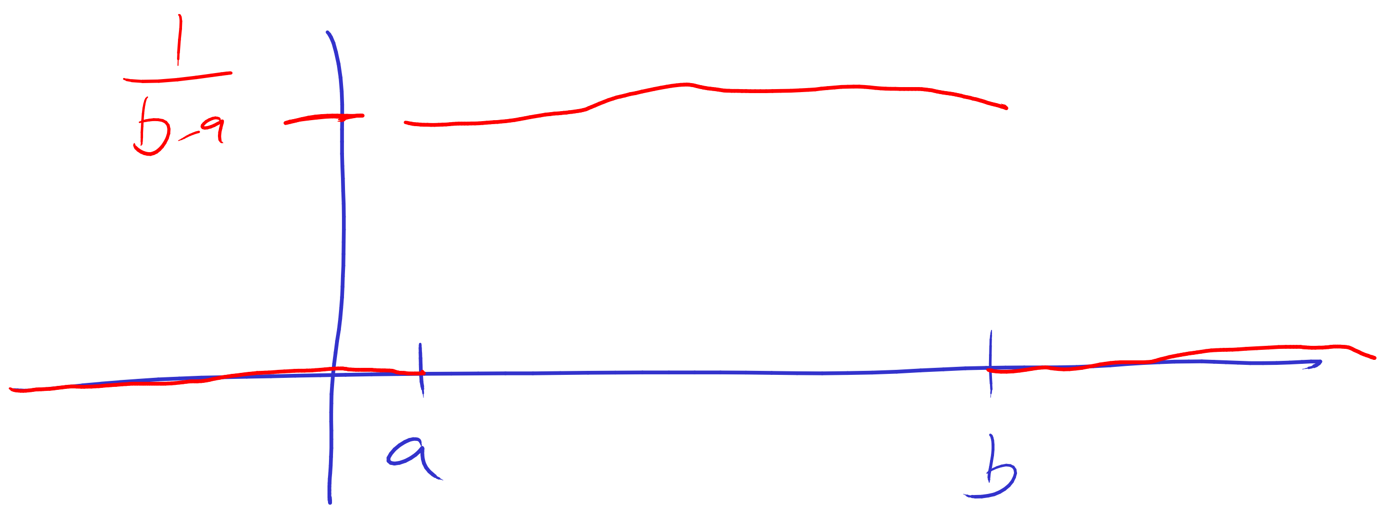
$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & ow \end{cases}$$



$$F(x) = \frac{1}{b-a} \quad a \le x \le b$$

$$E(x) = \int_a^b x \frac{1}{b-a} \, dx$$

$$\frac{1}{b-a} \int_a^b x \, dx$$

$$\frac{1}{b-a} \left[\frac{x^2}{2} \Big|_a^b \right]$$

$$\frac{1}{b-a} \left[\frac{b^2}{2} - \frac{a^2}{2} \right]$$

$$\frac{b^2 - a^2}{2(b-a)}$$

$$\frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}}$$

$$E(x) \quad \frac{b+a}{2}$$

Expected Value

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Normal Distribution

A random variable is said to follow a normal distribution if the the frequency of occurrence follow a Gaussian function.

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$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}$$

$$-\infty \leq X \leq \infty$$

Expected Value

$$E(x) = \mu$$

$$= \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{-\infty}^{\infty} x \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$x = z\sigma + \mu$$

$$z = \frac{x-\mu}{\sigma}$$

$$dz = dx/\sigma$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x \, e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sigma}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (z\sigma + \mu) \, e^{-\frac{z^2}{2}} \, dz$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\sigma \, e^{-\frac{z^2}{2}} \, dz + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mu \, e^{-z^2/2} \, dz$$

$$\mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-z^2/2} \, dz$$

1

$z \sim N(\mu = 0, \sigma^2 = 1)$

μ

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} z\,\sigma\, e^{-z^2/2}\, dz \qquad t = -z^2$$

$$dt = -2z\,dz$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sigma\, e^{t/2}\, \frac{dt}{-2} \quad \to \quad u$$

u

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Moments

The kth moment is defined as the expectation of the random variable, raised to the kth power, defined as $E(X^k)$.

Moment Generating Functions

The moment generating functions is used to obtain the kth moment. The mgf is defined as

$$m(t) = E(e^{tX})$$

The kth moment can be obtained by taking the kth derivative of the mgf, with respect to t, and setting t equal to 0:

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Characteristic Functions

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- Normal Distribution
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MGF

Expected Value

Variance

Variance

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- Normal Distribution
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MGF

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- Uniform Distribution
- Normal Distribution
- Moment Generating Functions
- Characteristic Functions
- Poisson Distribution
- Binomial Distribution
- **Uniform Distribution**
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- MGF Properties

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- Uniform Distribution
- Normal Distribution
- Moment Generating Functions
- Characteristic Functions
- Poisson Distribution
- Binomial Distribution
- Uniform Distribution
- **Normal Distribution**
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- **MGF Properties**

Linearity

Let X follow a distribution f, with the an MGF $M_X(t)$, the MGF of $Y = aX + b$ is given as

$$M_Y(t) = e^{tb} M_X(at)$$

Derivation

Linearity

Let X and Y be two random variables with MGFs $M_X(t)$ and $M_Y(t)$, respectively, and are independent. The MGF of $U = X - Y$

$$M_U(t) = M_X(t)M_Y(-t)$$

Derivation

Uniqueness

Let X and Y have the following distributions $F_X(x)$ and $F_Y(y)$ and MGFs $M_X(t)$ and $M_Y(t)$, respectively. X and Y have the same distribution $F_X(x) = F_Y(y)$ if and only if $M_X(t) = M_Y(t)$.

Uniqueness

Let $X_1, \cdots, X_n$ be independent random variables, where $X_i \sim N(\mu_i, \sigma_i^2)$, with $M_{X_i}(t) = \exp\{\mu_i t + \sigma_i^2 t^2/2\}$ for $i = 1, \cdots, n$. Find the MGF of $Y = a_1 X_1 + \cdots + a_n X_n$, where $a_1, \cdots, a_n$ are constants.