Review:

More Probability Theory

Moments

The kth moment is defined as the expectation of the random variable, raised to the kth power, defined as $E(X^k)$.

$$E(X^k) = \sum_{x} x^k P(X=x)$$

$$E(X^k) = \int_{x} x^k f(x)\,dx$$

The moment generating functions is used to obtain the kth moment. The mgf is defined as

$$m(t) = E(e^{tX})$$

The kth moment can be obtained by taking the kth derivative of the mgf, with respect to t, and setting t equal to 0:

$$E(X^k) = \left.\frac{d^k m(t)}{dt}\right|_{t=0}$$

$$M(t) = E(e^{tx})$$

$$e^{t} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!}$$

$$+ \cdots \cdots +$$

$$E(e^{tx}) = E\left(1 + tX + \frac{(tX)^2}{2!} + \right.$$

$$+ \frac{t^3 x^3}{3!} + \cdots \Big)$$

$$= E(1) + t\,E(x) + \frac{t^2 E(x^2)}{2!}$$

$$+ \frac{t^3 E(x^3)}{3!} + \cdots$$

Characteristic Functions

Characteristic Functions

$$\phi(t) = E\left(e^{itX}\right) = E\left\{\cos(tX)\right\} + iE\left\{\sin(tX)\right\}$$

Poisson Distribution

$$P(X=x) = \frac{e^{-\lambda} \lambda^x}{x!}, \quad x = 0 \ldots.$$

$$M(t) = E\left(e^{tX}\right)$$

$$m(t) = \sum_{x=0}^{\infty} e^{tx} \frac{e^{-\lambda} \lambda^x}{x!}$$

$$e^a = \sum_{x=0}^{\infty} \frac{a^x}{x!}$$

$$e^{tx} \qquad \lambda^{x}$$

$$e^{-\lambda} \sum_{x=0}^{\infty} \frac{e^{tx} \lambda^{x}}{x!} \qquad e^{a} = \sum_{x=0}^{\infty} \frac{a^{x}}{x!}$$

$$e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^{t} \lambda)^{x}}{x!} \qquad a = e^{t}\lambda$$

$$e^{-\lambda} e^{\lambda e^{t}}$$

$M(t) = e^{\lambda e^t - \lambda}$

Find $E(x)$?

$$E(x^1) = \left.\frac{d^1 M(t)}{dt^1}\right|_{t=0}$$

$$\left. e^{\lambda e^t - \lambda} \cdot \lambda e^t \right|_{t=0}$$

$$e^{\lambda(1) - \lambda} \cdot \lambda e^0 = \lambda$$

Expected Value

Variance

Binomial Distribution

$$P(X=x) = \binom{n}{x} p^x (1-p)^{n-x}$$

$$M(t) = E(e^{tx}) = \sum_{x=0}^{n} e^{tx} \binom{n}{x} p^x (1-p)^{n-x}$$

Binomial Theorem

$$(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$$

$$b = 1-p \qquad a = e^t p$$

$$\sum \binom{n}{x} e^{tx} p^x (1-p)^{n-x}$$

$$\sum \binom{n}{x} (e^t p)^x (1-p)^{n-x}$$

$$M(t) = (e^t p + 1 - p)^n$$

Uniform Distribution

MGF

Normal Distribution

$E(e^{tx})$ $\quad f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$

$-\infty \leq x \leq \infty$

① u-sub

② complete the square

$$E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}} dx$$

$$z = \frac{x-\mu}{\sigma}$$

$$\sigma z + \mu = x$$

$$dz = \frac{dx}{\sigma}$$

$$\frac{1}{\sqrt{2\pi}} \int e^{tx} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \frac{dx}{\sigma}$$

$$z = \frac{x-\mu}{\sigma}$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t(\sigma z + \mu)} \quad e^{-\frac{z^2}{2}} \quad dz$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{t\sigma z + t\mu - z^2/2} \quad dz$$

$$\frac{1}{\sqrt{2\pi}} e^{t\mu} \int_{-\infty}^{\infty} e^{t\sigma z - z^2/2}\, dz$$

$$\frac{1}{\sqrt{2\pi}} e^{t\mu} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(z^2 - 2t\sigma z)}\, dz$$

$$\int e^{-\frac{1}{2}(\underbrace{z^2 - 2zt\sigma + t^2\sigma^2} - t^2\sigma^2)}\, dz$$

$$\int e^{-\frac{1}{2}\left[(z - t\sigma)^2 - t^2\sigma^2\right]}\, dz$$

$$\frac{1}{\sqrt{2\pi}} e^{t\mu} \int e^{-\frac{(z-t\sigma)^2}{2}} e^{\frac{t^2\sigma^2}{2}} \, dz$$

$$\frac{1}{\sqrt{2\pi}} e^{t\mu + \frac{t^2\sigma^2}{2}} \int_{-\infty}^{\infty} e^{-\frac{(z-t\sigma)^2}{2}} \, dz$$

$$e^{t\mu + t^2\sigma^2/2} \int_{-\infty}^{\infty} \cancelto{1}{\frac{1}{\sqrt{2\pi}} e^{-\frac{(z-t\sigma)^2}{2}}} \, dz$$

$$z \sim N(t\sigma, 1)$$

$$e^{t\mu + t^2\sigma^2/2} = M(t)$$

MGF Properties

Linearity

Let X follow a distribution f, with the an MGF $M_X(t)$, the MGF of $Y = aX + b$ is given as

$$M_Y(t) = e^{tb} M_X(at)$$

Derivation

Linearity

Let X and Y be two random variables with MGFs $M_X(t)$ and $M_Y(t)$, respectively, and are independent. The MGF of $U = X - Y$

$$M_U(t) = M_X(t)M_Y(-t)$$

Derivation

Let X and Y have the following distributions $F_X(x)$ and $F_Y(y)$ and MGFs $M_X(t)$ and $M_Y(t)$, respectively. X and Y have the same distribution $F_X(x) = F_Y(y)$ if and only if $M_X(t) = M_Y(t)$.

Let $X_1, \cdots, X_n$ be independent random variables, where $X_i \sim N(\mu_i, \sigma_i^2)$, with $M_{X_i}(t) = \exp\{\mu_i t + \sigma_i^2 t^2/2\}$ for $i = 1, \cdots, n$. Find the MGF of $Y = a_1 X_1 + \cdots + a_n X_n$, where $a_1, \cdots, a_n$ are constants.

Function of Random Variables

Function of Random Variables

Obtaining the PDFs

Let there be a random variable X with a known distribution function $F_X(x)$, the density function for the random variable $Y = g(X)$ can be found with the following steps

1. Find the region of Y in the space of X, find $g^{-1}(y)$
2. Find the region of $Y \leq y$
3. Find $F_Y(y) = P(Y \leq y)$ using the probability density function of X over region $Y \leq y$
4. Find $f_Y(y)$ by differentiating $F_Y(y)$

Example 1

Let X have the following probability density function:

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 3X - 1$?

Let there be a random variable X with a known distribution function $F_X(x)$, if the random variable $Y = g(X)$ is either increasing or decreasing, than the probability density function can be found as

$$f_Y(y) = f_X\{g^{-1}(y)\}\left|\frac{dg^{-1}(y)}{dy}\right|$$

Example 2

Let X have the following probability density function:

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 + x & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 5 - (X/2)$?

Using the uniqueness property of Moment Generating Functions, for a random variable X with a known distribution function $F_X(x)$ and random variable $Y = g(X)$, the distribution of Y can be found by:

1. Find the moment generating function of Y, $M_Y(t)$.
2. Compare $M_Y(t)$, with known moment generating functions. If $M_Y(t) = M_V(t)$, for all values t, them Y and V have identical distributions.

Let X follow a normal distribution with mean μ and variance σ^2. Find the distribution of $Z = \frac{X-\mu}{\sigma}$.

Example 4

Let Z follow a standard normal distribution with mean 0 and variance 1. Find the distribution of $Y = Z^2$