Review:

More Probability Theory

Independent Random Variables

Let X and Y be independent random variables with joint density function $f_{XY}(x, y)$. Let $g(X)$ and $h(Y)$ be functions of X and Y, respectively.

Then:

$$E\{g(X)h(Y)\} = E\{g(X)\}\,E\{h(Y)\}$$

$$E(g(x)\,h(y)) = \int_Y \int_X g(x)\, h(y) f(x,y)\, dx\, dy$$

Because of independence

$$\int_Y \int_X g(x)\, \underline{h(y)}\, f_X(x)\, \underline{f_Y(y)}\, dx\, dy$$

$$\int_Y h(y)\, f_Y(y) \underbrace{\int_X g(x)\, f_X(x)\, dx}\, dy$$

$$\int_y h(y) f_Y(y) \underbrace{E(g(x))}_{x} \, dy$$

$$E(g(x)) \int_y h(y) f_Y(y) \, dy$$

$$E(g(x)) \, E(h(y))$$

Linearity

Let X follow a distribution f, with the an MGF $M_X(t)$, the MGF of $Y = aX + b$ is given as

$$M_Y(t) = e^{tb} M_X(at)$$

$$E(e^{tx}) = \int_x e^{tx} f(x)\, dx = M_x(t)$$

$$E(e^{ty}) = E(e^{t(ax+b)})$$

$$\int_x e^{t(ax+b)} f(x)\,dx$$

$$\int_x e^{tax} e^{bt} f(x)\,dx$$

$$e^{bt} \int_x e^{atx} f(x)\,dx \leftarrow t$$

$$e^{bt} M_X(at) = M_Y(t)$$

$$Y = aX + b$$

Linearity

Let X and Y be two random variables with MGFs $M_X(t)$ and $M_Y(t)$, respectively, and are independent. The MGF of $U = X - Y$

$$M_U(t) = M_X(t)M_Y(-t)$$

$$M_U(t) = E\left(e^{tU}\right) = E\left(e^{t(X-Y)}\right)$$

$$E\left(e^{tx} e^{-ty}\right)$$

$$E\left(e^{tx}\right) E\left(e^{-ty}\right)$$

$$M_x(t) \qquad M_y(-t)$$

Uniqueness

Let X and Y have the following distributions $F_X(x)$ and $F_Y(y)$ and MGFs $M_X(t)$ and $M_Y(t)$, respectively. X and Y have the same distribution $F_X(x) = F_Y(y)$ if and only if $M_X(t) = M_Y(t)$.

Uniqueness

Let $X_1, \cdots, X_n$ be independent random variables, where $X_i \sim N(\mu_i, \sigma_i^2)$, with $M_{X_i}(t) = \exp\{\mu_i t + \sigma_i^2 t^2/2\}$ for $i = 1, \cdots, n$. Find the MGF of $Y = a_1 X_1 + \cdots + a_n X_n$, where $a_1, \cdots, a_n$ are constants.

$X_1, \ldots, X_n \sim N(\mu_i, \sigma_i^2)$

$$M_{X_i}(t) = \exp\{ \mu_i t + \sigma_i^2 t^2/2 \}$$

$$Y = \sum_{i=1}^{n} a_i X_i \qquad M_Y(t)\,?$$

$$M_Y(t) = E(e^{tY}) = E\left(e^{t \sum_{i=1}^{n} a_i X_i}\right)$$

$$E\left(\prod_{i=1}^{n} e^{t a_i X_i}\right) = \prod_{i=1}^{n} E\left(e^{t a_i X_i}\right)$$

$$\prod_{i=1}^{n} M_{X_i}(a_i t) \qquad M_{X_i}(t) = \exp\{\mu_i t + \sigma_i^2 t^2/2\}$$

$$\prod_{i=1}^{n} \exp\left\{ \mu_i a_i t + \sigma_i^2 (a_i t)^2 / 2 \right\}$$

$$\prod_{i=1}^{n} \exp\left\{ \mu_i a_i t + \sigma_i^2 a_i^2 t^2 / 2 \right\}$$

$$\exp\left\{ \sum_{i=1}^{n} \left(\mu_i a_i t + \sigma_i^2 a_i^2 t^2 / 2 \right) \right\}$$

$$\exp\left\{ \sum_{i=1}^{n} \mu_i a_i t + \sum_{i=1}^{n} \sigma_i^2 a_i^2 t^2 / 2 \right\}$$

$$\exp\left\{ t \sum_{i=1}^{n} \mu_i a_i + \frac{t^2}{2} \sum_{i=1}^{n} \sigma_i^2 a_i^2 \right\}$$

$$\phi = \sum_{i=1}^{n} \mu_i a_i \qquad \tau^2 = \sum_{i=1}^{n} \sigma_i^2 a_i^2$$

$$\exp\left\{\phi t + \tau^2 \frac{t^2}{2}\right\} = M_Y(t)$$

$$Z \sim N(\mu, \sigma^2)$$

$$M_Z(t) = \exp\left\{\mu t + \sigma^2 t^2/2\right\}$$

$X_1, \ldots, X_n$ random sample

identical independentally distributed

as $N(\mu, \sigma^2)$

Find the distribution of

$$\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

MGF of $N(\mu, \sigma^2) = \exp\{\mu t + \sigma^2 t^2/2\}$

$$M_{\bar{X}}(t) = E\left(e^{t\bar{X}}\right) = E\left(e^{t\frac{1}{n}\sum X_i}\right)$$

$$E\left(e^{\Sigma \frac{1}{n} X_i t}\right) = E\left(\prod e^{\frac{1}{n} X_i t}\right)$$

$$\prod E\left(e^{1/n\, X_i t}\right) = \prod M_{X_i}(t/n)$$

$$M_{X_i}(t) = e^{\mu t + t^2\sigma^2/2}$$

$$\prod \exp\left\{\mu\, t/n + \sigma^2 (t/n)^2/2\right\}$$

$$\prod \exp\left\{ \mu t/n + \frac{\sigma^2 t^2}{2n^2} \right\}$$

$$\exp\left\{ \sum_{i=1}^{n} \left(\mu t/n + \frac{\sigma^2 t^2}{2n^2} \right) \right\}$$

$$\sum_{i=1}^{3} a X_i + b$$

$$a X_1 + b + a X_2 + b + a X_3 + b$$

$$a\sum_{i=1}^{3} X_i + 3b$$

$$\exp\left\{\sum_{i=1}^{n}\left(\mu t/n + \sigma^2 t^2/2n^2\right)\right\}$$

$$\exp\left\{\sum \mu t/n + \sum \sigma^2 t^2/2n^2\right\}$$

$$\exp\left\{\cancel{n}\mu t/\cancel{n} + \cancel{n}\sigma^2 t^2/2n^{\cancel{2}}\right\}$$

$$\exp\left\{\mu t + \sigma^2/n \; t^2/2\right\}$$

$\mu_{\bar{x}} = \mu \qquad \sigma^2_{\bar{x}} = \frac{\sigma^2}{n}$

$\bar{X} \sim N(\mu, \sigma^2/n)$

Using the Distribution Function

Let there be a random variable X with a known distribution function $F_X(x)$, the density function for the random variable $Y = g(X)$ can be found with the following steps

1. Find the region of Y in the space of X, find $g^{-1}(y)$
2. Find the region of $Y \leq y$
3. Find $F_Y(y) = P(Y \leq y)$ using the probability density function of X over region $Y \leq y$
4. Find $f_Y(y)$ by differentiating $F_Y(y)$

Example 1

Let X have the following probability density function:

$$f_X(x) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 3X - 1$?

Using the PDF

Let there be a random variable X with a known distribution function $F_X(x)$, if the random variable $Y = g(X)$ is either increasing or decreasing, than the probability density function can be found as

$$f_Y(y) = f_X\{g^{-1}(y)\} \left| \frac{dg^{-1}(y)}{dy} \right|$$

Example 2

Let X have the following probability density function:

$$f_X(x) = \begin{cases} \frac{3}{2}x^2 + x & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

Find the probability density function of $Y = 5 - (X/2)$?

Using the MGF

Using the uniqueness property of Moment Generating Functions, for a random variable X with a known distribution function $F_X(x)$ and random variable $Y = g(X)$, the distribution of Y can be found by:

1. Find the moment generating function of Y, $M_Y(t)$.
2. Compare $M_Y(t)$, with known moment generating functions. If $M_Y(t) = M_V(t)$, for all values t, them Y and V have identical distributions.

Example 3

Let X follow a normal distribution with mean μ and variance σ^2. Find the distribution of $Z = \frac{X-\mu}{\sigma}$.

Example 4

Let Z follow a standard normal distribution with mean 0 and variance 1. Find the distribution of $Y = Z^2$