# Joint Distribution Functions

# Conditional Distributions

A conditional distribution provides the probability of a random variable, given that it was conditioned on the value of a second random variable.

# Discrete Conditional Distributions

Let $X$ and $Y$ be 2 discrete random variables, with a joint distribution function of

$$p_{X,Y}(x,y) = P(X = x, Y = y)$$

The conditional distribution of $X|Y = y$ is defined as

$$p_{X|Y=y}(x) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

# Continuous Conditional Distributions

Let $X$ and $Y$ be 2 continuous random variables, with a joint density function of $f_{X,Y}(x, y)$. The conditional distribution of $X|Y = y$ is defined as

$$f_{X|Y=y}(x) = \frac{f_{X,Y}(x, y)}{f_Y(y)}$$

# Bivariae Normal Conditional Distribution

# Independent Random Variables

Random variables are considered independent of each other if the probability of one variable does not affect the probability of another variable.

# Discrete Independent Random Variables

Let $X$ and $Y$ be 2 discrete random variables, with a joint density function of $p_{X,Y}(x, y)$. $X$ is independent of $Y$ if and only if

$$p_{X,Y}(x, y) = p_X(x)p_Y(y)$$

# Continuous Independent Random Variables

Let $X$ and $Y$ be 2 continuous random variables, with a joint density function of $f_{X,Y}(x, y)$. $X$ is independent of $Y$ if and only if

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# Matrix Algebra

$$A = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$$

$$\det(A) = a_1 a_2$$

$$A^{-1} = \begin{pmatrix} 1/a_1 & 0 \\ 0 & 1/a_2 \end{pmatrix}$$

# Example

$$\begin{pmatrix} X \\ Y \end{pmatrix} \sim N\left\{ \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}, \begin{pmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix} \right\}$$

Show that $X \perp Y$.

$$f_{X,Y}(x,y) = \det(2\pi\Sigma)^{-1/2} \exp\left\{ -\frac{1}{2}(w-\mu)^T\Sigma^{-1}(w -$$

where $\Sigma = \begin{pmatrix} \sigma_y^2 & 0 \\ 0 & \sigma_y^2 \end{pmatrix}$, $\mu = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix}$, and

$$w = \begin{pmatrix} x \\ \end{pmatrix}$$

# Covariance

Let $X$ and $Y$ be 2 random variables with mean $E(X) = \mu_x$ and $E(Y) = \mu_y$, respectively. The covariance of $X$ and $Y$ is defined as

$$Cov(X, Y) = E\{(X - \mu_x)(Y - \mu_y)\}$$

$$Cov(X, Y) = E(XY) - \mu_x \mu_y$$

# Covariance

If $X$ and $Y$ are independent random variables, then

$$Cov(X, Y) = 0$$

# Correlation

The correlation of $X$ and $Y$ is defined as

$$\rho = Cor(X, Y) = \frac{Cov(X, Y)}{\sqrt{Var(X)Var(Y)}}$$

# Bivariate Normal Distribution

# Expectations

Let $X = (X_1, X_2, \dots, X_n)^{\mathrm{T}}$ be a set of random variables, the expectation of a function $g(X)$ is defined as

$$E\{g(X)\} = \sum_{x_1 \in X_1} \cdots \sum_{x_n \in X_n} g(X) p(x, \theta)$$

or

$$E\{g(X)\} = \int_{x_1 \in X_1} \cdots \int_{x_n \in X_n} g(X) f(x, \theta) dx_n \cdots dx_1$$

# Expected Value and Variance of Linear Functions

Let $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_m$ be random variables with $E(X_i) = \mu_i$ and $E(Y_j) = \tau_j$. Furthermore, let $U = \sum_{i=1}^{n} a_i X_i$ and $V = \sum_{j=1}^{m} b_j Y_j$ where $\{a_i\}_{i=1}^{n}$ and $\{b_j\}_{j=1}^{m}$ are constants. We have the following properties:

- $E(U) = \sum_{i=1}^{n} a_i \mu_i$
- $Var(U) = \sum_{i=1}^{n} a_i^2 Var(X_i) + 2 \sum\sum_{i<j} a_i a_j Cov(X_i, X_j)$
- $Cov(U, V) = \sum_{i=1}^{n} \sum_{j=1}^{m} Cov(X_i, Y_j)$

# Conditional Expectations

Let $X_1$ and $X_2$ be two random variables, the conditional expectation of $g(X_1)$, given $X_2 = x_2$, is defined as

$$E\{g(X_1)|X_2 = x_2\} = \sum_{x_1} g(x_1)p(x_1|x_2)$$

or

$$E\{g(X_1)|X_2 = x_2\} = \int_{x_1} g(x_1)f(x_1|x_2)dx_1.$$

# Conditional Expectations

Furthermore,

$$E(X_1) = E_{X_2}\{E_{X_1|X_2}(X_1|X_2)\}$$

and

$$Var(X_1) = E_{X_2}\{Var_{X_1|X_2}(X_1|X_2)\}+Var_{X_2}\{E_{X_1|X_2}$$