Method of Moment Estimator

Estimators

An *estimator* is an operation computing the value of an estimate, that targets the parameter, using measurements from a sample.

Data

Let $X_1, \dots, X_n \overset{iid}{\sim} F(\theta)$ where $F(\cdot)$ is a known distribution function and θ is a vector of parameters. Let $X = (X_1, \dots, X_n)^{\mathrm{T}}$, be the sample collected.

Method of Moments

Let the kth moment be defined as μ_k and the corresponding kth moment average $\frac{1}{n}\sum_{i=1}^{n} X_i^k$:

$$\mu_k = \frac{1}{n}\sum_{i=1}^{n} X_i^k.$$

The parameter estimates are for t parameters are the solutions for μ_k for $k = 1, \ldots, t$.

Bernoulli Distribution

Let $X_1, \ldots, X_n \overset{iid}{\sim} \mathrm{Bin}(1, p)$, find the method of moments estimator for p.

$$X_i \sim \mathrm{Bin}(n, p)$$

$$n = 1$$

$$E(X_i) = np$$

$$\hat{E}(X_i) = p$$

$$\tilde{p} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Poisson Distribution

Let $X_1, \dots, X_n \overset{iid}{\sim} \mathrm{Pois}(\lambda)$, find the method of moments estimator for λ.

$X_i \sim \mathrm{Pois}(\lambda)$

$$E(X_i) = \lambda$$

$$\tilde{\lambda} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

Uniform Distribution

Let $X_1, \dots, X_n \overset{iid}{\sim} U(1, \theta)$, find the method of moments estimator for θ.

$X_i \sim U(1, \theta)$

$$E(X_i) = \int_1^\theta \frac{1}{\theta - 1}\, dx$$

$$E(X_i) = \frac{\theta + 1}{2} = \frac{1}{n} \sum_{i=1}^{n} X_i$$

$$\hat{\theta} = 2\bar{X} - 1$$

Gamma Distribution

Let $X_1, \dots, X_n \overset{iid}{\sim} \mathrm{Gamma}(\alpha, \beta)$, find the method of moments estimator for α and β.

$$E(X_i) = \alpha\beta$$

$$Var(X_i) = \alpha\beta^2$$

$$E(X_i^2) = \alpha\beta^2 + \alpha^2\beta^2$$

$$\alpha\beta = \bar{X} \qquad \alpha = \frac{\bar{X}}{\beta}$$

$$\alpha\beta^2 + \alpha^2\beta^2 = \frac{1}{n}\sum X_i^2$$

$$\alpha\beta^2 + \bar{X}^2 = \frac{1}{n}\sum X_i^2$$

$$\alpha\beta^2 = \frac{1}{n}\sum X_i^2 - \bar{X}^2$$

$$\frac{\bar{X}}{\cancel{\beta}}\beta^{\cancel{2}} \to \tilde{\beta} = \frac{\frac{1}{n}\sum X_i^2 - \bar{X}^2}{\bar{X}}$$

$$\tilde{\alpha} = \frac{\bar{X}^2}{\frac{1}{n}\sum X_i^2 - \bar{X}^2}$$

Nomal Distribution

Let $X_1, \dots, X_n \overset{iid}{\sim} N(\mu, \sigma^2)$, find the method of moments estimator for μ and σ^2.

$X_i \sim N(\mu, \sigma^2)$

$E(X_i) = \mu$

$E(X_i^2) = Var(X_i) + E(X_i)^2$

$$Var(x_i) = E(x^2) - E(x_i)^2$$

$$E(x_i^2) = \sigma^2 + \mu^2$$

$$\mu = \frac{1}{n} \Sigma x_i$$

$$\sigma^2 + \mu^2 = \frac{1}{n} \Sigma x_i^2$$

$$\sigma^2 + \bar{x}^2 = \frac{1}{n} \Sigma x_i^2$$

$$\hat{\sigma}^2 = \frac{1}{n} \Sigma x_i^2 - \bar{x}^2$$