Test about a pop. mean. μ

Assumptions.

① $X = \{X_1, \dots, X_n\} \overset{iid}{\sim} N(\mu_0, \sigma^2)$

② Population standard deviation σ is known

We are interested to determine what values of μ are plausible from the data.

We conduct a hypothesis test on values that may be plausible.

$H_0: \mu = \mu_0$

$H_a: \mu < \mu_0$
$>$
$\neq$

① Construct the null & alt. hypothesis. And hypothesized value μ_0

②. Construct a test statistic that is appropriate to test the null hypothesis.

$$\bar{X} \sim N(\mu, \sigma^2/n)$$

$$\bar{X} - \mu \sim N(0, \sigma^2/n)$$

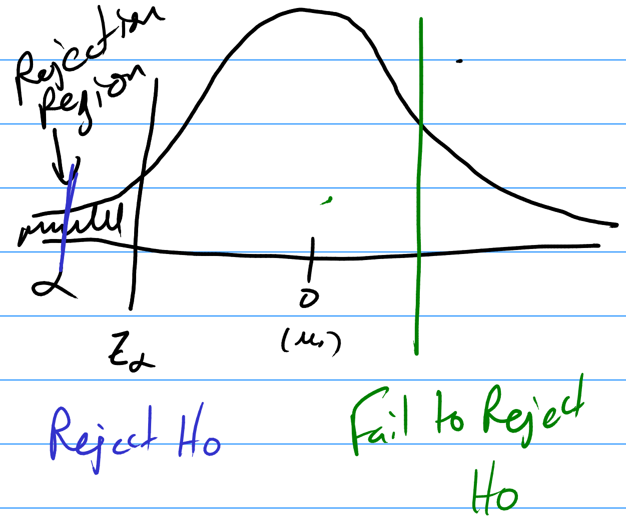
$$ts = \frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \sim N(0,1)$$

$H_0: \mu = \mu_0$

$$ts = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$ ← The distribution associated with the null hypothesis.

ts. ts.

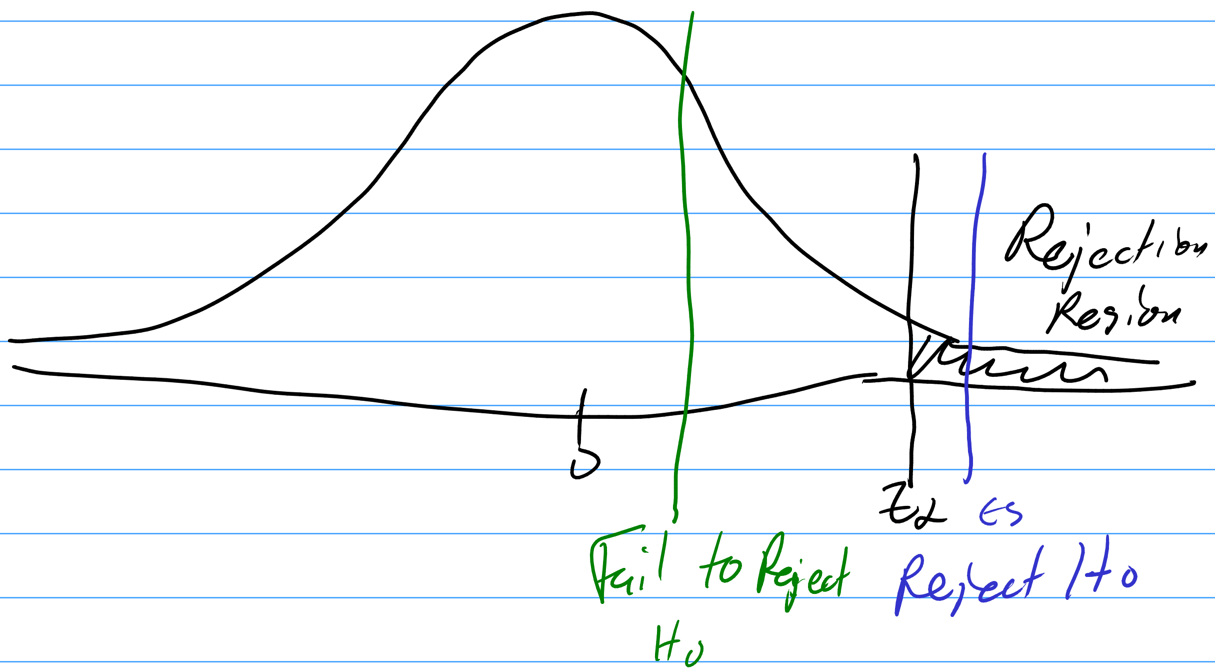
Ha: $\mu < \mu_0$; α - sig level



$$P(Z < ts) = p\text{-value}$$

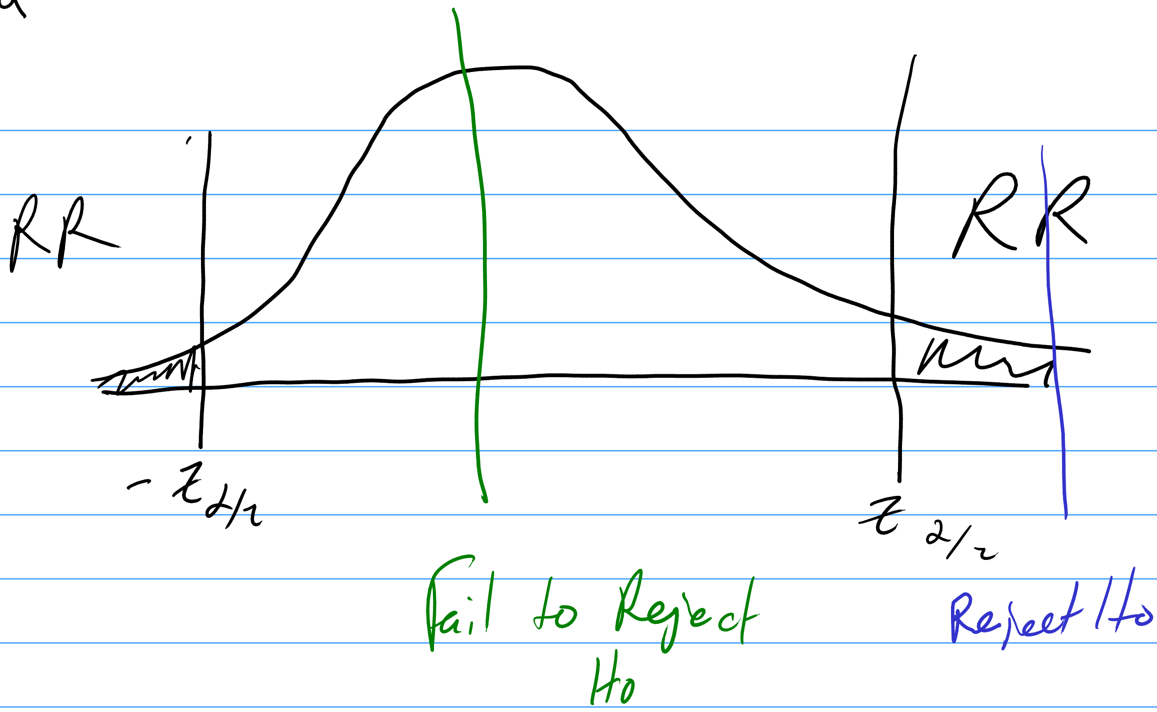
$p < \alpha$ Reject Ho

Ha $\mu > \mu_0$



$$P(Z > ts)$$

$H_a: \mu \neq \mu_0$



$$\text{p-value} = 2 \cdot P(Z > |ts|)$$

fire sprinkler company, they claim that the sprinkle usually go off when the room temp reach $130^\circ F$ on average. Conduct study on a sample of 9 sprinkler tests which yields an average temp activation of $131.08^\circ F$. It is known the pop. variation is $1.5^2 \,^\circ F$. Is the manufactures' claim true?

$H_0: \mu = 130^\circ F$
$H_a: \mu \neq 130^\circ F$

$$ts = \frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}}$$

$$= \frac{131.08 - 130}{1.5/3} \doteq 2.16$$

$2 \cdot P(z > |2.16|) = [1 - P(z < 2.16)] \cdot 2$

$= 1 - .9846$

$= 0.0154 \cdot 2$

p-value $= 0.0308$ $\quad \alpha = 0.05$

$p < \alpha$ Reject H_0

Type II Error probability of failing to reject H_0 given that it was false

CV - $H_0: \mu = \mu_0$ $\quad H_a: \mu > \mu_0$

$ts > z_\alpha$ Reject H_0

$$\frac{\bar{X} - \mu_0}{\sigma/\sqrt{n}} > z_\alpha$$

$$\bar{X} - \mu_0 > z_\alpha \sigma/\sqrt{n}$$

$$\bar{X} > \mu_0 + z_\alpha \sigma/\sqrt{n}$$

What if μ' is the true value

$\beta(\mu') = P(H_0$ fails to reject given that μ' is true$)$

$$= P(\bar{X} < \mu_0 + z_\alpha \sigma/\sqrt{n} \mid \mu = \mu')$$

$$= P\left(\frac{\bar{X} - \mu'}{\sigma/\sqrt{n}} < \frac{\mu_0 + z_\alpha \sigma/\sqrt{n} - \mu'}{\sigma/\sqrt{n}} \middle| \mu = \mu'\right)$$

$$= P\left(\frac{\bar{X} - \mu'}{\sigma/\sqrt{n}} < z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}} \middle| \mu = \mu'\right)$$

$\underbrace{Z^*}$ ← RV

$$P\left(Z^* < Z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}} \,\middle|\, \mu = \mu'\right)$$

$$\beta(\mu') = P\left(Z^* < Z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) \quad Z^* \sim N(0,1)$$

$$\text{Power} = 1 - \beta(\mu')$$

$H_0: \mu = 130 \qquad H_a: \mu > 130$

What is the sample size needed to determine that μ' is correct with a Type I error 0.05 Type II error of 0.1, where $\mu' = 131$.

$$Z_\alpha = 1.645$$

$$P^{-1}\left(P\left(Z < 1.645 + \frac{130 - 131}{1.5/\sqrt{n}}\right)\right) = P^{-1}(0.1)$$

$$Z_\beta = +1.28$$

$$-1.28 = 1.645 + \frac{-1}{1.5/\sqrt{n}}$$

$$-1.28 - 1.645 =$$

$$-0.365 \cdot 1.5 =$$

$$-0.5475 =$$

$$\left[\frac{1.5\,(1.645 + 1.28)}{-1}\right]^2$$

$$n = 19.25 \approx 20$$

$H_0\ \mu = \mu_0$ $\qquad\qquad$ $\phi(x) = P(Z < x)$

$\beta(\mu')$

$H_a: \mu > \mu_0$ $\qquad$ $\phi\left(Z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$

$H_a: \mu < \mu_0$ $\qquad$ $1 - \phi\left(-Z_\alpha + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$

$H_a: \mu \neq \mu_0$ $\qquad$ $\phi\left(Z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right) - \phi\left(-Z_{\alpha/2} + \frac{\mu_0 - \mu'}{\sigma/\sqrt{n}}\right)$

For α-level test with $\beta(\mu') = \beta$ and μ' as the alt.

$$n = \begin{cases} \left[\dfrac{\sigma(|Z_\alpha| + |Z_\beta|)}{\mu_0 - \mu'}\right]^2 & \text{one-tail tests. } \mu <> \mu_0 \\ \left[\dfrac{\sigma(|Z_{\alpha/2}| + |Z_\beta|)}{\mu_0 - \mu'}\right]^2 & \text{two-tail test } \mu \neq \mu_0 \end{cases}$$